HIGH SCHOOL

TRADITIONAL PLUS

MODEL COURSE SEQUENCE

Algebra I

Geometry

Algebra II

Achieve, Inc.

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Traditional Plus Course
Algebra I

This one-year Algebra I course is designed to build on a rigorous pre-algebra experience such as one indicated by the grade 8 expectations found in the National Assessment of Educational Progress (NAEP) guidelines and Achieve's Model Middle School Courses, as well as in many states' 8th grade standards.

In particular, it is expected that students will come to this course with a strong conceptual foundation in ratios, rates, and proportional relationships and an understanding of simple linear and non-linear patterns of growth and their representation in the coordinate plane. Facility with real number operations and integer exponents and roots is assumed. Students entering Algebra I with this set of knowledge and skills should be prepared to extend and deepen their understanding of numbers and their application. Algebra I has a strong emphasis on extending and formalizing understanding of advanced linear as well as simple exponential and quadratic functions and equations. To support this work, students in Algebra I will be expected to acquire more advanced number skills including facility with rational exponents and radical expressions.

This course is the first in a three-year traditional course sequence that has been enhanced by the addition of topics taken from data analysis, statistics and discrete mathematics. The algebraic reasoning and skills developed in this course will be applied to geometric contexts in the second traditional course and will form the foundation for the work done in the third course, where more advanced functions are introduced and generalizations across all function families more fully explored.

Appropriate use of technology is expected in all work in this Traditional Plus course sequence. In Algebra I, this includes employing technological tools to assist students in the formation and testing of conjectures, creating graphs and data displays, and determining and assessing lines of fit for data. Testing with and without technological tools is recommended.

How a particular subject is taught influences not only the depth and retention of the content of a course but also the development of skills in inquiry, problem solving, and critical thinking. Every opportunity should be taken to link the concepts of Algebra I to those encountered in middle school mathematics as well as to other disciplines. Students should be encouraged to be creative and innovative in their approach to problems, to be productive and persistent in seeking solutions, and to use multiple means to communicate their insights and understanding.

The Major Concepts below provide the focus for the Algebra I course. They should be taught using a variety of methods and applications so that students attain a deep understanding of these concepts. Maintenance Concepts should have been taught previously and are important foundational concepts that will be applied in this Algebra I course. Continued facility with and understanding of the Maintenance Concepts is essential for success in the Major Concepts defined for Algebra I.

MAJOR CONCEPTS

- Applications of Numbers
- Mathematical Vocabulary and Logic
- Properties of Irrational and Real Numbers
- Patterns of Growth Through Iteration
- Linear, Proportional, and Piecewise-Linear Functions
- Linear Equations, Inequalities and Systems of Linear Equations
- Exponents and Simple Exponential Functions
- Quadratic Functions and Equations Over the Real Numbers

MAINTENANCE CONCEPTS

- Real Numbers Operations
- Whole Number Exponents and Roots
- Ratios, Rates, and Proportions
- Functions and Graphing in the Coordinate Plane
- Linear Equations and Inequalities and Their Applications
- Linear and Simple Exponential Patterns of Growth
- Elementary Data Analysis

A. Applications of Numbers

This course begins by reinforcing student understanding of real numbers and their multiple representations. Measurement situations, including those that employ derived measures and estimation, provide opportunities for students to apply real numbers. Scientific and factorial notation extends students' repertoire of mathematical representations to facilitate the communication of the extremely large and small numbers that often arise in applications.

Successful students will:

A1 Extend and apply understanding about rates and ratios, estimation, and measurement to derived measures including weighted averages; use appropriate units and unit analysis to express and check solutions.

Derived measures are those achieved through calculations with measurements that can be taken directly.

a. Identify applications that can be expressed using derived measures or weighted averages.

Examples: Percent change and density are examples of derived measures; grade averages, stock market indexes, the consumer price index, and unemployment rates are examples of weighted averages.

A2 Interpret, compare, and use extreme numbers involving significant figures, orders of magnitude, and scientific notation; determine a reasonable degree of precision when making calculations or estimations.

a. Identify applications that involve extreme numbers.

Examples: Extreme numbers include lottery odds, national debt, astronomical distances, or the size of a connector in a microchip.

b. Assess the amount of error resulting from estimation and determine whether the error is within acceptable tolerance limits.

A3 Determine the number of ways events can occur using permutations, combinations, and other systematic counting methods.

A permutation is a rearrangement of distinct items in which their order matters; a combination is a selection of a given number of distinct items from a larger number without regard to their arrangement (i.e., in which their order does not matter).

a. Know and apply organized counting techniques such as the Fundamental Counting Principle.

The Fundamental Counting Principal is a way of determining the number of ways a sequence of events can take place. If there are n ways of choosing one thing and m ways of choosing a second after the first has been chosen, then the Fundamental Counting Principal says that the total number of choice patterns is $n \bullet m$.

Examples: How many different license plates can be formed with two letters and three numerals? If the letters had to come first, how many letters would be needed to create at least as many different license plate numbers? How many different subsets are possible for a set with six elements?

b. Distinguish between counting situations that do not permit replacement and situations that do permit replacement.

Examples: How many different four-digit numbers can be formed if the first digit must be non-zero and each digit may be used only once? How many are possible if the first digit must be non-zero but digits can be used any number of times?

c. Distinguish between situations where order matters and situations where it does not; select and apply appropriate means of computing the number of possible arrangements of the items in each case.

A4 Interpret and simplify expressions involving factorial notation; recognize simple cases where factorial notation may be used to express a result.

Examples: Interpret 6! as the product $6 \bullet 5 \bullet 4 \bullet 3 \bullet 2 \bullet 1$; recognize that $\frac{15!}{12!} = 15 \bullet 14 \bullet 13 = 2{,}730$; express the number of possible orders in which 7 names can be listed on a ballot as 7!.

B. Mathematical Vocabulary and Logic

The deductive reasoning that leads from assumptions and definitions to an irrefutable conclusion sets mathematics apart from the inductive reasoning found in science or the situational reasoning found in most other disciplines. The grounding for such logical thought and expression is a solid conceptual understanding of sets and their operations—also important as the theoretical foundation for digital systems and for describing logic circuits and formulating search engine queries—knowledge and careful application of mathematical vocabulary and notation and attention to mathematical syntax. The mathematical content encountered earlier in middle school courses as well as common media outlets provide excellent resources through which students can learn to carefully analyze and interpret statements, explore generalizations, and formulate, test, critique, and verify or refute conjectures.

Successful students will:

B1 Know the concepts of sets, elements, empty set, relations (e.g., belong to), and subsets, and use them to represent relationships among objects and sets of objects.

a. Recognize and use different methods to define sets (lists, defining property).

b. Perform operations on sets: union, intersection, complement.

Example: Use Boolean search techniques to refine online bibliographic searches.

c. Create and interpret Venn diagrams to solve problems.

d. Identify whether a given set is finite or infinite; give examples of both finite and infinite sets.

B2 Use and interpret mathematical notation, terminology, and syntax.

a. Use and interpret common mathematical terminology.

Examples: Use and interpret conjunctions, disjunctions, and negations ("and," "or," "not"); use and interpret terms of causation ("if…then") and equivalence ("if and only if").

b. Describe logical statements using terms such as *assumption, hypothesis, conclusion, converse*, and *contraposition*.

c. Recognize uses of logical terms in everyday language, noting the similarities and differences between common use and their use in mathematics.

B3 Analyze and apply logical reasoning.

a. Distinguish between deductive and inductive reasoning; identify the strengths of each type of reasoning and its application in mathematics.

Inductive reasoning is based on observed patterns and can be used in mathematics to generate conjectures after which deductive reasoning can be used *to show that the conjectures are true in all circumstances. Inductive reasoning cannot prove propositions; valid conclusions and proof require deduction.*

b. Explain and illustrate the importance of generalization in mathematics and its relationship to inductive and deductive reasoning.

c. Make, test, and confirm or refute conjectures using a variety of methods; construct simple logical arguments and proofs; determine simple counterexamples.

d. Recognize syllogisms, tautologies, and circular reasoning and use them to assess the validity of an argument.

e. Recognize and identify flaws or gaps in the reasoning used to support an argument.

Example: Recognize that $A \Rightarrow B$ does not imply that $B \Rightarrow A$.

f. Explain reasoning in both oral and written forms.

B4 Analyze and apply algorithms for searching, for sorting, and for solving optimization problems.

a. Identify and apply algorithms for searching, such as sequential and binary.

b. Describe and compare simple algorithms for sorting, such as bubble sort, quick sort, and bin sort.

Example: Compare strategies for alphabetizing a long list of words; describe a process for systematically solving the Tower of Hanoi problem.

c. Know and apply simple optimization algorithms.

Example: Use a vertex-edge graph (network diagram) to determine the shortest path needed to accomplish some task.

C. Properties of Irrational and Real Numbers

Building on the understanding of whole number exponents, simple roots, and the relationship between the square root and the exponent $\frac{1}{2}$, students will develop an understanding of the impact of a negative exponent and generalize the properties of exponents to all rational exponents. Facility with both exponential and radical expression of numbers opens up the ability to compute in either form and to deepen student insight into the real number system. Properties such as the density of the rational numbers can be explored on an informal level, through example and explanation rather than formal proof, and lead naturally to the ability to estimate irrational numbers to any given degree of precision.

Successful students will:

C1 Interpret and apply integral and rational exponents in numerical expressions.

a. Use rational exponents to rewrite numerical expressions.

Examples: $64^{\frac{5}{6}} = (64^{\frac{1}{6}})^5 = 2^5 = 32$; $3^{-2} = \frac{1}{9}$; $5^{\frac{3}{2}} = \sqrt{5^3} = 5\sqrt{5}$.

b. Convert between forms of numerical expressions involving roots; perform operations on numbers expressed in exponential or radical form.

Examples: Convert $\sqrt{8}$ to $2\sqrt{2}$ and use the understanding of this conversion to perform similar calculations and to compute with numbers in radical form; rewrite $\frac{2}{5^{-4}} \bullet \frac{3^{-3}}{5^2} \bullet \frac{3^5}{2^5}$ as a fraction having only positive exponents and as a fraction in lowest terms. $\frac{2}{5^{-4}} \bullet \frac{3^{-3}}{5^2} \bullet \frac{3^5}{2^5} = \frac{2 \bullet 3^5 \bullet 5^4}{3^3 \bullet 5^2 \bullet 2^5} = \frac{2 \bullet 3^2 \bullet 5^2}{2^5} = \frac{225}{16}$.

C2 Establish simple facts about rational and irrational numbers using logical arguments and examples.

Examples: Explain why, if *r* and *s* are rational, then both *r* + *s* and *rs* are rational (for example, both $\frac{3}{4}$ and 2.3 are rational in $\frac{3}{4} + 2.3 = \frac{3}{4} + \frac{23}{10} = \frac{15}{20} + \frac{46}{20} = \frac{61}{20}$, which is the ratio of two integers, hence rational); give examples to show that, if *r* and *s* are irrational, then *r* + *s* and *rs* could be either rational or irrational (for example, $\sqrt{3} + \frac{\sqrt{3}}{2}$ is irrational whereas $\left(5 + \sqrt{2}\right) - \sqrt{2}$ is rational); show that a given interval on the real number line, no matter how small, contains both rational and irrational numbers.

C3 Given a degree of precision, determine a rational approximation to that degree of precision for an irrational number expressed using rational exponents or radicals.

D. Patterns of Growth Through Iteration

This is an opportunity to deepen and extend understanding of linear and simple exponential patterns of growth studied in middle school. Sequences can be represented on a coordinate plane and the characteristics of the resulting graphs tied to previously encountered concepts, such as slope.

Successful students will:

D1 Analyze, interpret, and describe relationships represented iteratively and recursively, including those produced using a spreadsheet.

Examples: Recognize that the sequence defined by "First term = 5. Each term after the first is six more than the preceding term" is the sequence whose first seven terms are 5, 11, 17, 23, 29, 35, and 41.; recognize that the result of repeatedly squaring a number between −1 and 1 appears to approach zero, while the result of repeatedly squaring a number less than −1 or greater than 1 appears to continue to increase; determine empirically how many steps are needed to produce four-digit accuracy in square roots by iterating the operations divide and average.

D2 Generate and describe sequences having specific characteristics; use calculators and spreadsheets effectively to extend sequences beyond a relatively small number of terms.

a. Generate and describe the factorial function or the Fibonacci sequence recursively.

b. Generate and describe *arithmetic* sequences recursively; identify arithmetic sequences expressed recursively.

Arithmetic sequences are those in which each term differs from its preceding term by a constant difference. To describe an arithmetic sequence, both the starting term and the constant difference must be specified.

Example: $a_1 = 5$, $a_{n+1} = a_n + 2$ describes the arithmetic sequence 5, 7, 9, 11, . . .

c. Generate and describe *geometric* sequences recursively; identify geometric sequences expressed recursively.

Geometric sequences are those in which each term is a constant multiple of the term that precedes it. To describe a geometric sequence both the starting term and the constant multiplier (often called the common ratio) must be specified.

Example: $a_1 = 3$, $a_{n+1} = -2a_n$ describes the geometric sequence 3, -6, 12, -24, . . .

d. Given an irrational number expressed using rational exponents or radicals, find increasing and decreasing sequences that converge to that number and show that the first terms of these sequences satisfy the right inequalities.

Example: $1 < 1.4 < 1.41 < 1.414 < \ldots < \sqrt{2} < 1.415 < 1.42 < 1.5 < 2$ since

$$1^2=1 < (1.4)^2=1.96 < (1.41)^2=1.9881 < (1.414)^2=1.999396 < \ldots < \left(\sqrt{2}\right)^2=2$$

$$< \ldots < (1.415)^2=2.002225 < (1.42)^2=2.0164 < (1.5)^2=2.25 < 2^2=4$$

D3 Demonstrate the effect of compound interest, exponential decay, or exponential growth using iteration.

Examples: Using a spreadsheet, enter the amount of a loan, the monthly interest rate, and the monthly payment in a spreadsheet. The formula (loan amount) • (1+interest rate) – (monthly payment) gives the amount remaining monthly on the loan at the end of the first month, and the iterative “fill down” command will show the amount remaining on the loan at the end of each successive month; a similar process using past data about the yearly percent increase of college tuition and annual inflation rate will provide an estimate of the cost of college for a newborn in current dollar equivalents.

a. Identify the diminishing effect of increasing the number of times per year that interest is compounded and relate this to the notion of instantaneous compounding.

E. Linear, Proportional, and Piecewise-Linear Functions

Linear patterns of growth are a focus of the middle school curriculum. Description, analysis, and interpretation of lines should continue to be reinforced and extended as students work with linear functions. The reciprocal functions introduced here should be linked back to student experience with proportions and with the simple exponential patterns of growth studied in middle school. Absolute value, step, and other piecewise-linear functions are included to extend student facility with linear growth to situations that are defined differently over subsets of their domains, a fairly common phenomenon in real-life contexts. Also included here is a first look at how changes in parameters affect the graph of a function.

Successful students will:

E1 Recognize, graph, and use direct proportional relationships.

A proportion is composed of two pairs of real numbers, (a, b) and (c, d), with at least one member of each pair non-zero, such that both pairs represent the same ratio. A linear function in which f(0) = 0 represents a direct proportional relationship. The function f(x) = kx, where k is constant, describes a direct proportional relationship.

a. Analyze the graph of direct proportional relationships, *f(x) = kx,* and identify its key characteristics.

   A direct proportional relationship is represented by a line that passes through the origin (0, 0) whose slope is the constant of proportionality.

b. Compare and contrast the graphs of x = k, y = k, and y = kx, where k is a constant.

c. Recognize and provide a logical argument that if *f(x)* is a linear function, *g(x) = f(x)* – *f*(0) represents a direct proportional relationship.

   Example: If *f(x)* is linear, *f(x) = mx + b* and f(0) = 0 + *b,* so *g(x) = f(x)* – *f*(0) = *mx + b – b = mx*. This means that *g*(0) = 0, so the function *y = g(x)* is a direct proportional relationship.

d. Recognize quantities that are directly proportional and express their relationship symbolically.

   Example: The relationship between length of the side of a square and its perimeter is directly proportional.

E2 Recognize, graph, and use reciprocal relationships.

A function of the form f(x) = k/x where k is constant describes a reciprocal relationship. The term "inversely proportional" is sometimes used to identify such relationships, however, this term can be very confusing since the word "inverse" is also used in the term "inverse function" (the function $y = f^{-1}(x)$ *with the property that* $f \circ f^{-1}(x) = f^{-1} \circ f(x) = x$*, which describes the identity function).*

a. Analyze the graph of reciprocal relationships, *f(x) = k/x,* and identify its key characteristics.

   The graph of f(x) = k/x is not a straight line and does not cross either the x– or the y–axis (i.e., there is no value of x for which f(x) = 0, nor is there any value for f(x) if x= 0). It is a curve consisting of two disconnected branches, called a hyperbola.

b. Recognize quantities that are inversely proportional and express their relationship symbolically.

Example: The relationship between lengths of the base and side of a rectangle with fixed area is inversely proportional.

E3 Distinguish among and apply linear, direct proportional, and reciprocal relationships; identify and distinguish among applications that can be expressed using these relationships.

a. Identify whether a table, graph, formula, or context suggests a linear, direct proportional, or reciprocal relationship.

b. Create graphs of linear, direct proportional, and reciprocal functions by hand and using technology.

c. Distinguish practical situations that can be represented by linear, direct proportional, or reciprocal relationships; analyze and use the characteristics of these relationships to answer questions about the situation.

E4 Explain and illustrate the effect of varying the parameters *m* and *b* in the family of linear functions, $f(x) = mx + b$, and varying the parameter *k* in the families of direct proportional and reciprocal functions, $f\left(x\right) = kx$ and $f\left(x\right) = \frac{k}{x}$, respectively.

E5 Identify key characteristics of absolute value, step, and other piecewise-linear functions and graph them.

a. Interpret the algebraic representation of a piecewise-linear function; graph it over the appropriate domain.

b. Write an algebraic representation for a given piecewise-linear function.

c. Determine vertex, slope of each branch, intercepts, and end behavior of an absolute value graph.

d. Recognize and solve problems that can be modeled using absolute value, step, and other piecewise-linear functions.

Examples: Postage rates, cellular telephone charges, tax rates.

F. Linear Equations, Inequalities and Systems of Linear Equations

Just as with linear functions, solving linear equations and inequalities in one variable should be reinforced as necessary before extending those skills to literal equations, equations, and inequalities involving absolute values, linear inequalities in two variables, and systems of equations and inequalities.

Successful students will:

F1 Solve linear and simple nonlinear equations involving several variables for one variable in terms of the others.

Example: Solve $A = \pi r^2 h$ for h or for r.

F2 Solve equations and inequalities involving the absolute value of a linear expression in one variable.

F3 Graph the solution of linear inequalities in two variables.

a. Know what it means to be a solution of a linear inequality in two variables, represent solutions algebraically and graphically, and provide examples of ordered pairs that lie in the solution set.

b. Graph a linear inequality in two variables and explain why the graph is always a half-plane (open or closed).

F4 Solve systems of linear equations in two variables using algebraic procedures.

F5 Create, interpret, and apply mathematical models to solve problems arising from contextual situations that involve linear relationships.

a. Distinguish relevant from irrelevant information, identify missing information, and find what is needed or make appropriate estimates.

b. Apply problem solving heuristics to practical problems: Represent and analyze the situation using symbols, graphs, tables, or diagrams; assess special cases; consider analogous situations; evaluate progress; check the reasonableness of results; and devise independent ways of verifying results.

c. Recognize and solve problems that can be modeled using linear inequalities in two variables or a system of linear equations in two variables; interpret the solution(s) in terms of the context of the problem.

Examples: Time/rate/distance problems; problems involving percentage increase or decrease; break-even problems, such as those comparing costs of two services; optimization problems that can be approached through linear programming.

d. Represent linear relationships using tables, graphs, verbal statements and symbolic forms; translate among these forms to extract information about the relationship.

F6 Determine, interpret, and compare linear models for data that exhibits a linear trend.

a. Identify and evaluate methods of determining the goodness of fit of a linear model.

Examples: A linear model might pass through the most points, minimize the sum of the absolute deviations, or minimize the sum of the square of the deviations.

b. Use a computer or a graphing calculator to determine a linear regression equation (least-squares line) as a model for data that suggests a linear trend.

c. Determine and interpret correlation coefficients.

d. Use and interpret residual plots to assess the goodness of fit of a regression line.

e. Note the effect of outliers on the position and slope of the regression line.

f. Interpret the slope and y-intercept of the regression line in the context of the relationship being modeled.

G. Exponents and Simple Exponential Functions

The work done earlier with numerical expressions involving exponents is extended here to expressions involving variables. Variable expressions involving negative integral exponents are linked to students' earlier work with reciprocal functions. Once solid understanding of the properties of exponents is established, students are introduced to exponential functions, that is, to functions in which the independent variable appears in the exponent. While students will have encountered applications involving exponential behavior as early as middle school, in this course students will be expected to gain facility over the symbolic and graphic representation of such relationships in simple cases. Algebra I exponential functions, along with reciprocal functions, provide students with non-linear examples through which they can come to a deeper understanding of linear behavior. More complex exponential applications, the solution of exponential equations, and the relationship between exponential and logarithmic functions will be addressed in Algebra II.

Successful students will:

G1 Apply the properties of exponents to transform variable expressions involving integral and rational exponents.

a. Translate between rational exponents and notation involving integral powers and roots.

Examples: $a^p \cdot a^q = a^{p+q}$; $\frac{x^5}{x^7} = \frac{1}{x^2} = x^{-2}$; $9^x = 3^{2x}$; $\left(8b^6\right)^{\frac{1}{3}} = 2b^2$; $x^{\frac{4}{5}} = \sqrt[5]{x^4} = \left(\sqrt[5]{x}\right)^4$.

b. Factor out common factors with exponents.

Examples: $6v^7 + 12v^5 - 8v^3 = 2v^3(3v^4 + 6v^2 - 4)$; $3x(x + 1)^2 - 2(x + 1)^2 = (x + 1)^2(3x - 2)$.

"Chunking" is a term often used to describe treating an expression, such as the $x + 1$ above as a single entity.

G2 Graph and analyze exponential functions and identify their key characteristics.

a. Identify functions having the general form $f(x) = ab^x + c$ for $b > 0$, $b \neq 1$ as exponential functions.

b. Recognize and represent the graphs of exponential functions; identify upper or lower limits (asymptotes).

G3 Recognize problems that can be modeled using exponential functions; interpret the solution(s) in terms of the context of the problem.

Exponential functions model situations where change is proportional to the initial quantity.

a. Use exponential functions to represent growth functions such as $f(x) = an^x$ ($a > 0$ and $n > 1$) and decay functions such as $f(x) = an^{-x}$ ($a > 0$ and $n > 1$).

b. Use the laws of exponents to determine exact solutions for problems involving exponential functions where possible; otherwise approximate the solutions graphically or numerically.

H. Quadratic Functions and Equations Over the Real Numbers

The final topic in Algebra I introduces yet another class of functions, those involving quadratic behavior. While students may have already encountered the classic application of quadratic functions, behavior of an object under the influence of gravity, Algebra I should present them with their first experience with general quadratic phenomena. This early work with quadratic functions rests on understanding and solving equations and should focus on those with real zeros/roots.

Successful students will:

H1 Identify quadratic functions expressed in multiple forms; identify the specific information each form clarifies.

a. Express a quadratic function as a polynomial, $f(x) = ax^2 + bx + c$, where a, b, and c are constants with $a \neq 0$, and identify its graph as a parabola that opens up when $a > 0$ and down when $a < 0$; relate c to where the graph of the function crosses the y-axis.

b. Express a quadratic function having integral roots in factored form, $f(x) = (x - r)(x - s)$, when r and s are integers; relate the factors to the solutions of the equation $(x - r)(x - s) = 0$ ($x = r$ *and* $x = s$) and to the points ((r, 0) and (s, 0)) where the graph of the function crosses the x-axis.

H2 Graph quadratic functions and use the graph to help locate zeros.

A zero of a quadratic function $f(x) = ax^2 + bx + c$ is a value of x for which $f(x) = 0$.

a. Sketch graphs of quadratic functions using both a graphing calculator and tables of values.

b. Estimate the real zeros of a quadratic function from its graph and identify quadratic functions that do not have real zeros by the behavior of their graphs.

H3 Recognize contexts in which quadratic models are appropriate; determine and interpret quadratic models that describe quadratic phenomena.

Examples: The relationship between length of the side of a square and its area; the relationship between time and distance traveled for a falling object.

H4 Solve quadratic equations with integral solutions; use quadratic equations to represent and solve problems involving quadratic behavior.

a. Solve quadratic equations that can be easily transformed into the form $(x - a)(x - b) = 0$ or $(x + a)^2 = b$, for integers a and b.

b. Estimate the roots of a quadratic equation from the graph of the corresponding function.

H5 Rewrite quadratic functions and interpret their graphical forms.

a. Write a quadratic function in polynomial or standard form, $f(x) = ax^2 + bx + c$, to identify the y-intercept of the function's parabolic graph or the x-coordinate of its vertex, $x = -\frac{b}{2a}$.

b. Write a quadratic function in factored form, $f(x) = a(x - r)(x - s)$, to identify its roots.

c. Write a quadratic function in vertex form, $f(x) = a(x - h)^2 + k$, to identify the vertex and axis of symmetry of the function's parabolic graph.

d. Determine domain and range, intercepts, axis of symmetry, and maximum or minimum.

H6 Graph quadratic equations and solve those with real solutions using a variety of methods.

a. Solve quadratic equations having real solutions by factoring, by completing the square, and by using the quadratic formula.

b. Use a calculator to approximate the roots of a quadratic equation and as an aid in graphing.

c. Select and explain a method of solution (e.g., exact vs. approximate) that is effective and appropriate to a given problem.

d. Recognize and solve practical problems that can be expressed using quadratic equations having real solutions; interpret their solutions in terms of the context of the situation.

Examples: Determine the height of an object above the ground t seconds after it has been thrown upward from a platform d feet above the ground at an initial velocity of v_0 feet per second; find the area of a rectangle with perimeter 120 in terms of the length, L, of one side.

H7 Make regular fluent use of basic algebraic identities such as $(a + b)^2 = a^2 + 2ab + b^2$; $(a - b)^2 = a^2 - 2ab + b^2$; and $(a + b)(a - b) = a^2 - b^2$.

a. Use the distributive law to derive each of these formulas.

Examples: $(a + b)(a - b) = (a + b)a - (a + b)b = (a^2 + ab) - (ab + b^2) = a^2 + ab - ab - b^2 = a^2 - b^2$; applying this to specific numbers, $37 \bullet 43 = (40 - 3)(40 + 3) = 1{,}600 - 9 = 1{,}591$.

b. Use geometric constructions to illustrate these formulas.

Example: Use a partitioned square or tiles to provide a geometric representation of $(a + b)^2 = a^2 + 2ab + b^2$.

Traditional Plus Course
Geometry

This Geometry course builds on geometry and measurement concepts students would have encountered in a strong middle school curriculum and extends some concepts addressed in Algebra I—such as coordinate geometry and the use of algebraic techniques—to analyze geometric relationships. It focuses on reasoning about geometric figures and their properties. Students are expected to make conjectures, prove theorems, and find counterexamples to refute false claims. They are expected to use properties and theorems to construct geometric objects and to perform transformations in the coordinate plane. An optional enrichment unit extending the study of three-dimensional objects to the geometry of a sphere is included for study as time permits. The habits and tools of analysis and logical reasoning developed through studying geometric topics can and should be applied throughout mathematics. With this in mind, the closing unit in this course applies these tools to probability and probability distributions.

Appropriate use of technology is expected in all work. In Geometry, this includes employing technological tools to assist students in the formation and testing of conjectures and in creating diagrams, graphs, and data displays. Geometric constructions should be performed using geometric software as well as classical tools, and technology should be used to aid three-dimensional visualization. Testing with and without technological tools is recommended.

How a particular subject is taught influences not only the depth and retention of the content of a course but also the development of skills in inquiry, problem solving, and critical thinking. Every opportunity should be taken to link the concepts of Geometry to those encountered in middle school and Algebra I as well as to other disciplines. Students should be encouraged to be creative and innovative in their approach to problems, to be productive and persistent in seeking solutions, and to use multiple means to communicate their insights and understanding.

The Major Concepts below provide the focus for the Geometry course. They should be taught using a variety of methods and applications so that students attain a deep understanding of these concepts. Maintenance Concepts should have been taught previously and are important foundational concepts that will be applied in this Geometry course. Continued facility with and understanding of the Maintenance Concepts is essential for success in the Major Concepts defined for Geometry.

MAJOR CONCEPTS

- Geometric Representations
- Reasoning in Geometric Situations
- Similarity, Congruence, and Right Triangle Trigonometry
- Circles
- Three-Dimensional Geometry
- Probability and Probability Distributions

MAINTENANCE CONCEPTS

- Geometric Objects and Their Properties
- Length, Area, and Volume
- Graphing in the Coordinate Plane
- Rigid Motions in the Coordinate Plane
- Proportional Reasoning and Similarity
- Two- and Three-Dimensional Representations
- Elementary Data Analysis
- Simple Probability

A. Geometric Representations

Students who have experienced a rigorous middle school mathematics curriculum will be familiar with geometric objects and their measurements and properties. Rather than review these topics directly, this course begins the year with problems involving angles, polygons, circles, solids, length, area, and volume set in the context of probability. It extends the representational repertoire by introducing discrete graphs that can also serve as a vehicle for clarifying real-life situations.

Successful students will:

A1 Recognize probability problems that can be represented by geometric diagrams, on the number line, or in the coordinate plane; represent such situations geometrically and apply geometric properties of length or area to calculate the probabilities.

> Example: What is the probability that three randomly chosen points on the plane are the vertices of an obtuse triangle?

A2 Use coordinates and algebraic techniques to interpret, represent, and verify geometric relationships.

> Examples: Given the coordinates of the vertices of a quadrilateral, determine whether it is a parallelogram; given a line segment in the coordinate plane whose endpoints are known, determine its length, midpoint, and slope; given the coordinates of three vertices of a parallelogram, determine all possible coordinates for the fourth vertex.

a. Interpret and use locus definitions to generate two- and three-dimensional geometric objects.

> Examples: The locus of points in the plane equidistant from two fixed points is the perpendicular bisector of the line segment joining them; the parabola defined as

the locus of points equidistant from the point (5, 1) and the line y = −5 is $y = \frac{1}{12}(x-5)^2 - 2$.

A3 Construct and interpret visual discrete graphs and charts to represent contextual situations.

a. Construct and interpret network graphs and use them to diagram social and organizational networks.

A graph is a collection of points (nodes) and the lines (edges) that connect some subset of those points; a cycle on a graph is a closed loop created by a subset of edges. A directed graph is one with one-way arrows as edges.

Examples: Determine the shortest route for recycling trucks; schedule when contestants play each other in a tournament; illustrate all possible travel routes that include four cities; interpret a directed graph to determine the result of a tournament.

b. Construct and interpret decision trees to represent the possible outcomes of independent events.

A tree is a connected graph containing no closed loops (cycles).

Examples: Classification of quadrilaterals, repeated tossing of a coin, possible outcomes of moves in a game.

c. Construct and interpret flow charts.

B. Reasoning in Geometric Situations

The geometric objects studied in a rigorous middle school curriculum provide a context for the development of deductive reasoning while also providing a context for students to explore properties, using inductive methods to propose and test conjectures. Students are expected to build on the understanding and formal language of reasoning introduced in Algebra I and use it to describe and justify geometric constructions and theorems.

Successful students will:

B1 Use the vocabulary of logic to describe geometric statements and the relationships among them.

a. Identify assumption, hypothesis, conclusion, converse, and contraposition for geometric statements.

b. Explain and illustrate the role of definitions, conjectures, theorems, proofs, and counterexamples in mathematical reasoning; use geometric examples to illustrate these concepts.

B2 Apply logic to assess the validity of geometric arguments.

a. Analyze the consequences of using alternative definitions for geometric objects.

b. Use geometric examples to demonstrate the effect that changing an assumption has on the validity of a conclusion.

c. Make, test, and confirm or refute geometric conjectures using a variety of methods.

d. Demonstrate through example or explanation how indirect reasoning can be used to establish a claim.

e. Present and analyze direct and indirect geometric proofs using paragraphs or two-column or flow-chart formats; explain how indirect reasoning can be used to establish a claim.

   Example: Explain why, if two lines are intersected by a third line in such a way as to make the corresponding angles, alternate interior angles, or alternate exterior angles congruent, then the two original lines must be parallel.

B3 Analyze, execute, explain, and apply simple geometric constructions.

a. Apply the properties of geometric figures and mathematical reasoning to perform and justify basic geometric constructions.

b. Perform and explain simple straightedge and compass constructions.

   Examples: Copy a line segment, an angle, and plane figures; bisect an angle; construct the midpoint and perpendicular bisector of a segment.

c. Use geometric computer or calculator packages to create and test conjectures about geometric properties or relationships.

C. Similarity, Congruence and Right Triangle Trigonometry

Building on the understanding of rates, ratios, and proportions developed in middle school and applied to number and algebraic situations in Algebra I, the geometric concepts of similarity and congruence are established in this course along with the conditions under which triangles, and later other polygons, are similar or congruent. Students are expected to prove theorems about the similarity of triangles or congruence of angles and triangles using various formats (paragraph, flow-chart, or two-column chart). Applications of similarity include origin-centered dilations in the coordinate plane and right triangle trigonometry.

Successful students will:

C1 Identify and apply conditions that are sufficient to guarantee similarity of triangles.

Informally, two geometric objects in the plane are similar if they have the same shape. More formally, having the same shape means that one figure can be mapped onto the other by means of rigid transformations and/or an origin-centered dilation.

a. Identify two triangles as similar if the ratios of the lengths of corresponding sides are equal (SSS criterion), if the ratios of the lengths of two pairs of corresponding sides and the measures of the corresponding angles between them are equal (SAS criterion), or if the measures of two pairs of corresponding angles are equal (AA criterion).

b. Apply the definition and characteristics of similarity to verify basic properties of angles and triangles and to perform using straightedge and compass or geometric software.

c. Identify the constant of proportionality and determine the measures of corresponding sides and angles for similar triangles.

d. Use similar triangles to demonstrate that the rate of change (slope) associated with any two points on a line is a constant.

e. Recognize, use, and explain why a line drawn inside a triangle parallel to one side forms a smaller triangle similar to the original one.

C2 Identify congruence as a special case of similarity; determine and apply conditions that guarantee congruence of triangles.

Informally, two figures in the plane are congruent if they have the same size and shape. More formally, having the same size and shape means that one figure can be mapped into the other by means of a sequence of rigid transformations.

a. Determine whether two plane figures are congruent by showing whether they coincide when superimposed by means of a sequence of rigid motions (translation, reflection, or rotation).

b. Identify two triangles as congruent if the lengths of corresponding sides are equal (SSS criterion), if the lengths of two pairs of corresponding sides and the measures of the corresponding angles between them are equal (SAS criterion), or if the measure of two pairs of corresponding angles and the length of the side that joins them are equal (ASA criterion).

c. Apply the definition and characteristics of congruence to make constructions, solve problems, and verify basic properties of angles and triangles.

Examples: Verify that the bisector of the angle opposite the base of an isosceles triangle is the perpendicular bisector of the base; construct an isosceles triangle with a given base angle.

C3 Extend the concepts of similarity and congruence to other polygons in the plane.

a. Identify two polygons as similar if have the same number of sides and angles, if corresponding angles have the same measure, and if corresponding sides are proportional; identify two polygons as congruent if they are similar and their constant of proportionality equals 1.

b. Determine whether or not two polygons are similar.

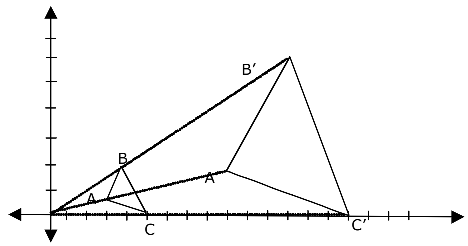
c. Use examples to show that analogues of the SSS, SAS, and AA criteria for similarity of triangles do not work for polygons with more than three sides.

C4 Analyze, interpret, and represent origin-centered dilations

An origin-centered dilation with scale factor r maps every point (x, y) in the coordinate plane to the point (rx, ry).

a. Interpret and represent origin-centered dilations of objects in the coordinate plane.

Example: In the following figure, triangle A'B'C' with A'(9,3), B'(12,6), and C'(15,0) is the dilation of triangle ABC with A(3,1), B(4,2), and C(5,0). The scale factor for this dilation is 3.



b. Explain why the image under an origin-centered dilation is similar to the original figure.

c. Show that an origin-centered dilation maps a line to a line with the same slope and that dilations map parallel lines to parallel lines (lines passing through the origin remain unchanged and are parallel to themselves).

C5 Show how similarity of right triangles allows the trigonometric functions sine, cosine, and tangent to be properly defined as ratios of sides.

a. Know the definitions of sine, cosine, and tangent as ratios of sides in a right triangle and use trigonometry to calculate the length of sides, measure of angles, and area of a triangle.

b. Derive, interpret, and use the identity $\sin^2\theta + \cos^2\theta = 1$ for angles θ between $0°$ and $90°$.

This identity is a special representation of the Pythagorean theorem.

D. Circles

Once students are familiar with methods of proof and have mastered theorems associated with angles, triangles and other polygons, they are ready to turn their efforts to recognizing and establishing the relationships among the lines, angles, arcs and areas associated with a circle. Application of these relationships to contextual situations as well as to diagrams is expected in this course.

Successful students will:

D1 Recognize and apply the definitions and the properties of a circle; verify relationships associated with a circle.

a. Know and apply the definitions of radius, diameter, chord, tangent, secant, and circumference of a circle.

b. Recognize and apply the fact that a tangent to a circle is perpendicular to the radius at the point of tangency.

c. Recognize, verify, and apply the relationships between central angles, inscribed angles, and circumscribed angles and the arcs they define.

Example: Show that a triangle inscribed on the diameter of a circle is a right triangle.

d. Recognize, verify, and apply the relationships between inscribed and circumscribed angles of a circle and the arcs and segments they define.

Example: Prove that if a radius of a circle is perpendicular to a chord of the circle, then it bisects the chord.

D2 Determine the length of line segments and arcs, the measure of angles, and the area of shapes that they define in complex geometric drawings.

Examples: Determine the amount of glass in a semi-circular transom; identify the coverage of an overlapping circular pattern of irrigation; determine the length of the line of sight on the earth’s surface.

D3 Relate the equation of a circle to its characteristics and its graph.

Examples: Determine the equation of a circle given its center and radius, and conversely, given the equation, determine its center and radius.

E. Three-Dimensional Geometry

It is important for students to develop sound spatial sense for the three-dimensional world in which we live. Volume and surface area of basic solids addressed earlier in middle school will be expanded here to include the calculation of slant height. Beyond the determination of volume and surface area, students should be able to identify the intersection of two or more planes or the figure formed when a solid object is cut by a plane (a cross-sectional slice). Visualization of a three-dimensional object should be extended from the interpretation of nets and multiple views, to the identification and analysis of a solid formed by rotating a line or curve around a given axis, a skill useful in architecture, carpentry, and the study of calculus. An optional topic in three-dimensional geometry extends the study of three dimensions to the exploration of geometry on a sphere, a non-Euclidean space where all lines intersect (that is where the parallel postulate from Euclidean geometry does not hold).

Successful students will:

E1 Determine surface area and volume of solids when slant height is not given; recognize and use relationships among volumes of common solids.

a. Determine the surface area and volume of spheres and of right prisms, pyramids, cylinders, and cones when slant height is not given.

b. Recognize and apply the 3:2:1 relationship between the volumes of circular cylinders, hemispheres, and cones of the same height and circular base.

c. Recognize that the volume of a pyramid is one-third the volume of a prism of the same base area and height and use this to solve problems involving such measurements.

E2 Analyze cross-sections of basic three-dimensional objects and identify the resulting shapes.

Example: Describe all possible results of the intersection of a plane with a cube, prism, pyramid, or sphere.

E3 Describe the characteristics of the three-dimensional object traced out when a one- or two-dimensional figure is rotated about an axis.

E4 Analyze all possible relationships among two or three planes in space and identify their intersections.

a. Know that two distinct planes will either be parallel or will intersect in a line.

b. Demonstrate that three distinct planes may be parallel; two of them may be parallel to each other and intersect with the third, resulting in two parallel lines; or none may be parallel, in which case the three planes intersect in a single point or a single line, or by pairs in three parallel lines.

E5. Recognize that there are geometries other than Euclidean geometry in which the parallel postulate is not true.

E6 Analyze and interpret geometry on a sphere.

OPTIONAL ENRICHMENT UNIT

a. Know and apply the definition of a great circle.

A great circle of a sphere is the circle formed by the intersection of the sphere with the plane defined by any two distinct, non-diametrically opposite points on the sphere and the center of the sphere.

Example: Show that arcs of great circles subtending angles of 180 degrees or less provide shortest routes between points on the surface of a sphere.

Since the earth is nearly spherical, this method is used to determine distance between distant points on the earth.

b. Use latitude, longitude, and great circles to solve problems relating to position, distance, and displacement on the earth's surface.

Displacement is the change in position of an object and takes into account both the distance and direction it has moved.

Example: Given the latitudes and longitudes of two points on the surface of the Earth, find the distance between them along a great circle and the bearing from one point to the other.

Bearing is the direction or angle from one point to the other relative to North = 0°. A bearing of N31°E means that the second point is 31° East of a line pointing due North of the first point.

c. Interpret various two-dimensional representations for the surface of a sphere, called projections (e.g., two-dimensional maps of the Earth), called projections, and explain their characteristics.

Common projections are Mercator (and other cylindrical projections), Orthographic and Stereographic (and other Azimuthal projections), pseudo-cylindrical, and sinusoidal. Each projection has advantages for certain purposes and has its own limitations and drawbacks.

d. Describe geometry on a sphere as an example of a non-Euclidean geometry in which any two lines intersect.

In spherical geometry, great circles are the counterpart of lines in Euclidean geometry. All great circles intersect. The angles between two great circles are the angles formed by the intersecting planes defined by the great circles.

Examples: Show that on a sphere, parallel lines intersect—that is, the parallel postulate does not hold true in this context; identify and interpret the intersection of lines of latitude with lines of longitude on a globe; recognize that the sum of the degree measures of the interior angles of a triangle on a sphere is greater than 180°.

F. Probability and Probability Distributions

The reasoning that is so integral to this course can and should be applied to other areas of mathematics as well as to life experiences. One common type of reasoning is statistical reasoning, which has its grounding in probability distributions. Just as we opened this course by looking at geometric settings for probability, we close it with an introductory look at using probability to help make informed decisions.

Successful students will:

F1 Calculate and apply probabilities of compound events.

a. Employ Venn diagrams to summarize information concerning compound events.

b. Distinguish between dependent and independent events.

c. Use probability to interpret odds and risks and recognize common misconceptions.

Examples: After a fair coin has come up heads four times in a row, explain why the probability of tails is still 50% in the next toss; analyze the risks associated with a particular accident, illness, or course of treatment; assess the odds of winning the lottery or being selected in a random drawing.

d. Show how a two-way frequency table can be used effectively to calculate and study relationships among probabilities for two events.

F2 Recognize and interpret probability distributions.

a. Identify and distinguish between discrete and continuous probability distributions.

b. Reason from empirical distributions of data to make assumptions about their underlying theoretical distributions.

c. Know and use the chief characteristics of the normal distribution

The normal (or Gaussian) distribution is actually a family of mathematical functions that are symmetric in shape with scores more concentrated in the middle than in the tails. They are sometimes described as bell shaped. Normal distributions may have differing centers (means) and scale (standard deviation). The standard normal distribution is the normal distribution with a mean of zero and a standard deviation of one. In normal distributions, approximately 68% of the data lie within one standard deviation of the mean and 95% within two.

Example: Demonstrate that the mean and standard deviation of a normal distribution can vary independently of each other (e.g., that two normal distributions with the same mean can have different standard deviations).

d. Identify common examples that fit the normal distribution (height, weight) and examples that do not (salaries, housing prices, size of cities) and explain the distinguishing characteristics of each.

e. Calculate and use the mean and standard deviation to describe characteristics of a distribution.

f. Understand how to calculate and interpret the expected value of a random variable having a discrete probability distribution.

F3 Apply probability to practical situations to make informed decisions.

Examples: Evaluate medical test results and treatment options, analyze risk in situations where anecdotal evidence is provided, interpret media reports and evaluate conclusions.

a. Communicate an understanding of the inverse relation of risk and return.

b. Explain the benefits of diversifying risk.

Traditional Plus Course
Algebra II

This Algebra II course describes the third year in a three-year traditional course sequence that has been enhanced by the addition of topics taken from data analysis, statistics, and discrete mathematics. It deepens and extends the understanding of linear, direct proportional, reciprocal, exponential, and quadratic relationships whose study was begun in earlier courses, encouraging students to view them as classes of functions and equations and objects of study in their own right. The study of linear phenomena, begun in middle school and expanded in Algebra I, serves as a starting point for this course. The introduction of complex numbers opens the door to the understanding and solution of *all* quadratic equations and their related functions. Power, root, polynomial, and rational functions; expressions; and equations increase student experience with non-linear behavior and its representation. In this course, students are asked to compare and contrast the properties of all of these different algebraic forms. They are expected to relate changes in the algebraic structure of each function to transformations of its graphical representation and are expected to recognize and solve problems that can be modeled using this range of functions. An optional enrichment unit on operations on functions introduces function composition and permits the definition of logarithms. This course can be enhanced even further by including another optional enrichment unit in the area of statistical studies and models, addressing such topics as transforming data and identifying a class of functions from among those studied that can be used to model data.

Throughout Algebra II, technology is an important tool for visualization and for deepening understanding of function relationships and transformations. Data utilities are essential for the optional enrichment unit that addresses data transformations. Testing with and without technological tools is recommended.

How a particular subject is taught influences not only the depth and retention of the content of a course but also the development of skills in inquiry, problem solving, and critical thinking. Every opportunity should be taken to link the concepts of Algebra II to those encountered in earlier mathematics courses as well as to other disciplines. Students should be encouraged to be creative and innovative in their approach to problems, to be productive and persistent in seeking solutions and to use multiple means to communicate their insights and understanding.

The Major Concepts below provide the focus for this Algebra II course. They should be taught using a variety of methods and applications so that students attain a deep understanding of these concepts. Maintenance Concepts should have been taught previously and are important foundational concepts that will be applied in this Algebra II course. Continued facility with and understanding of the Maintenance Concepts is essential for success in the Major Concepts defined for Algebra II.

MAJOR CONCEPTS

- Systems of Linear Equations and Inequalities
- Extending the Number System
- Quadratic Functions, Equations, and Inequalities
- Polynomial Functions, Expressions, and Equations
- Radical and Rational Functions and Equations
- Function Prototypes and Transformations
- Function Operations and Logarithms [OPTIONAL ENRICHMENT UNIT]
- Statistical Studies and Models

MAINTENANCE CONCEPTS

- Real Numbers and their Applications
- Rational Exponents and Radicals
- Linear Functions, Equations, and Inequalities
- Systems of Linear Equations
- Simple Exponential Relationships
- Quadratic Functions and Equations with Real Zeros/Roots

A. Systems of Linear Equations and Inequalities

Students who have completed a rigorous Algebra I course should be very familiar with solving systems of linear equations in two variables. While linear behavior was addressed in Geometry as well, this Algebra II unit provides an opportunity to reinforce that knowledge by extending it to systems of inequalities in two variables and to systems of equations in three variables. It also serves to reinforce understanding of linear behavior in general and extend it to lines in space.

Successful students will:

A1 Solve systems of two or more linear inequalities in two variables and graph the solution set.

> Example: The set of points (x, y) that satisfy all three inequalities $5x - y \geq 3$, $3x + y \leq 10$, and $4x - 3y \leq 6$ is a triangle; the solution to the system of inequalities $x + y + z \geq -10$, $2x - y + 3z \leq 20$, $8x - 2y + z \geq -3$ is the intersection of three half-planes whose points satisfy each inequality separately.

A2 Solve systems of linear equations in three variables using algebraic procedures; describe the possible arrangements of their graphs.

a. Relate the possible arrangements of the graphs of three linear equations in three variables to the number of solutions of the corresponding system of equations.

A3 Recognize and solve problems that can be modeled using a system of linear equations or inequalities; interpret the solution(s) in terms of the context of the problem.

Examples: Break-even problems, such as those comparing costs of two services; optimization problems that can be approached through linear programming.

B. Extending the Number System

While all linear equations and inequalities and systems of linear equations and inequalities can be solved within the real number system, solution of some higher-degree equations requires an extension of the real numbers to the complex number system. Students should understand that each successive number system from the natural numbers to the integers, rational numbers, real numbers, and complex numbers is contained in, or embedded in, the succeeding system. However, students should also understand that for complex numbers, a new form of number (*a + bi, for a and b real numbers)* requiring the use of two real numbers must be defined and that different operations of addition and multiplication must be defined in order to accommodate this new form. The complex number system maintains some, but not all, of the properties of the real number system.

Successful students will:

B1 Identify expressions of the form *a* + *bi* as complex numbers.

The imaginary unit, sometimes represented as $i = \sqrt{-1}$*, is a solution to the equation* $x^2 = -1$*.*

a. Explain why every real number is a complex number.

Every real number, a, is a complex number because it can be expressed as a + 0i.

b. Express the square root of a negative number in the form *bi* or 0 + *bi,* where *b* is real.

Just as with square roots of positive numbers, there are two square roots for negative numbers; in $\sqrt{-4} = \pm 2i$*, 2i is taken to be the principal square root based on both the Cartesian and trigonometric representations of complex numbers.*

Examples: Determine the principal square root for each of the following: $\sqrt{-7} = i\sqrt{7}$ or $0 + i\sqrt{7}$; $\sqrt{-256} = 16i$ or $0 + 16i$.

c. Identify complex conjugates.

The conjugate of a complex number *a* + *bi* is the number *a* – *bi*.

B2 Compute with complex numbers. [OPTIONAL ENRICHMENT UNIT]

a. Add, subtract, and multiply complex numbers using the rules of arithmetic.

b. Use conjugates to divide complex numbers.

Example: $\frac{(5+4i)}{3-2i} = \frac{(5+4i)}{(3-2i)} \bullet \frac{(3+2i)}{(3+2i)} = \frac{15+22i+8i^2}{9-4i^2} = \frac{7+22i}{13}$ or $\frac{7}{13} + \frac{22}{13}i$.

This process can also be applied to the division of irrational numbers involving square roots such as $a + \sqrt{b}$ and $a - \sqrt{b}$.

C. Quadratic Functions, Equations, and Inequalities

In Algebra I, work was limited to quadratic equations and functions for which there were real roots or zeros. Now that students have the system of complex numbers in which to work, they can apply the algebraic and graphical techniques learned earlier to *all* quadratic equations and inequalities. They should be able to predict the nature of the roots of a quadratic equation from its discriminant, identify characteristics of the solutions and coefficients of a quadratic equation in two variables from its graph, and relate the factored form of a quadratic to its solutions even when those solutions are not real. The usefulness of quadratic functions in expressing important real-world relationships should be reinforced with students now able to interpret complex as well as real solutions when they arise.

Successful students will:

C1 Solve quadratic equations over the complex numbers.

a. Use the quadratic formula or completing the square to solve any quadratic equation in one variable and write it as a product of linear factors.

b. Use the quadratic formula to show that the *x*-coordinate (abscissa) of the vertex of the corresponding parabola is halfway between the roots of the equation.

c. Use the discriminant $D = b^2 - 4ac$, to determine the nature of the roots of the equation $ax^2 + bx + c = 0$.

d. Identify quadratic functions that do not have real zeros by the behavior of their graphs.

   A quadratic function that does not cross the horizontal axis has no real zeros.

e. Show that complex roots of a quadratic equation having real coefficients occur in conjugate pairs; show that multiplying factors related to conjugate pairs results in a quadratic equation having real coefficients.

   Example: The complex numbers $\left(3 + i\sqrt{5}\right)$ and $\left(3 - i\sqrt{5}\right)$ are the roots of the equation $\left(x - \left(3 + i\sqrt{5}\right)\right)\left(x - \left(3 - i\sqrt{5}\right)\right) = x^2 - 6x + 14 = 0$ whose coefficients are real.

C2 Manipulate simple quadratic equations or functions to extract information.

a. Describe the effect that changes in the the leading coefficient or constant term of $f(x) = ax^2 + bx + c$ have on the shape, position, and characteristics of the graph of $f(x)$; identify the relationship of such transformations to the type of solutions of the equation.

   Examples: If *a* and *c* have opposite signs, then the roots of the quadratic must be real and have opposite signs; varying *c* varies the *y*-intercept of the graph of the parabola; if *a* is positive, the parabola opens up, if *a* is negative, it opens down; as $|a|$ increases, the graph of the parabola is stretched vertically, i.e., it looks narrower.

   Use completing the square to determine the center and radius of a circle.

C3 Solve and graph quadratic inequalities in one or two variables.

Example: Solve $(x - 5)(x + 1) > 0$ and relate the solution to the graph of $(x-5)(x + 1) > y$.

D. Polynomial Functions, Expressions, and Equations

To extend the graphing of quadratic functions, power functions form a natural bridge into the study of polynomial functions and expressions. The development of facility with algebraic expressions involving polynomials, including the development of the binomial expansion theorem and its connections to probability and combinatorics, is included here, along with understanding about how leading coefficients and constant terms of a polynomial contribute to characteristics of its graph.

Successful students will:

D1 Analyze power functions and identify their key characteristics.

In this course, a power function is any function defined over the real numbers of the form $f(x) = ax^p$ where p is a rational number. Power functions include positive integer power functions such as $f(x) = -3x^4$, root functions such as $f(x) = \sqrt{x}$ and $f(x) = 4x^{\frac{1}{3}}$, and reciprocal functions such as $f(x) = kx^{-4}$.

a. Recognize that the inverse proportional function $f(x) = k/x$ ($f(x) = kx^n$ for $n = -1$) and the direct proportional function $f(x) = kx$ ($f(x) = kx^n$ for $n = 1$) are special cases of power functions.

b. Distinguish between *odd* and *even* power functions.

Examples: When the exponent of a power function is a positive integer, then even power functions have either a minimum or maximum value, while odd power functions have neither; even power functions have reflective symmetry over the y-axis, while odd power functions demonstrate rotational symmetry about the origin.

c. Transform the algebraic expression of power functions using properties of exponents and roots.

Example: $f(x) = 3x^2\left(-2x^{-\frac{3}{2}}\right)$ can be more easily identified as a root function once it is rewritten as $f(x) = -6x^{\frac{1}{2}} = -6\sqrt{x}$.

d. Explain and illustrate the effect that a change in a parameter has on a power function (a change in a or n for $f(x) = ax^n$).

D2 Analyze polynomial functions and identify their key characteristics.

a. Know that polynomial functions of degree n have the general form $f(x) = ax^n + bx^{n-1} + \ldots + px^2 + qx + r$ for n an integer, $n \geq 0$ and $a \neq 0$.

The degree of the polynomial function is the largest power of its terms for which the coefficient is non-zero.

b. Know that a power function with an exponent that is a positive integer is a particular type of polynomial function, called a *monomial*, whose graph contains the origin.

c. Distinguish among polynomial functions of low degree, i.e., constant functions, linear functions, quadratic functions, or cubic functions.

d. Explain why every polynomial function of odd degree has at least one zero; identify any assumptions that contribute to your argument.

At this level, students are expected to recognize that this result requires that polynomials are connected functions without "holes." They are not expected to give a formal proof of this result.

e. Communicate understanding of the concept of the multiplicity of a root of a polynomial equation and its relationship to the graph of the related polynomial function.

If a root, r_1, of a polynomial function has multiplicity 3, $(x - r_1)^3$ is a factor of the polynomial. The graph of the polynomial touches the horizontal axis at r_1 but does not change sign (does not cross the axis) if the multiplicity of r_1 is even; it changes sign (crosses over the axis) if the multiplicity is odd.

D3 Use key characteristics to identify the graphs of simple polynomial functions.

Simple polynomial functions include constant functions, linear functions, quadratic functions or cubic functions such as $f(x) = x^3$, $f(x) = x^3 - a$, or $f(x) = x(x - a)(x + b)$.

a. Decide if a given graph or table of values suggests a simple polynomial function.

b. Distinguish between the graphs of simple polynomial functions.

c. Where possible, determine the domain, range, intercepts, and end behavior of polynomial functions.

The end behavior of a graph refers to the trend of the values of the function as x approaches $\pm\infty$. It should be noted that it is not always possible to determine exact x-intercepts. Graphing utilities are excellent vehicles for providing indications of end behavior and approximations for intercepts.

D4 Recognize and solve problems that can be modeled using power or polynomial functions; interpret the solution(s) in terms of the context of the problem.

a. Use power or polynomial functions to represent quantities arising from numeric or geometric contexts such as length, area, and volume.

Examples: The number of diagonals of a polygon as a function of the number of sides; the areas of simple plane figures as functions of their linear dimensions; the surface areas of simple three-dimensional solids as functions of their linear dimensions; the sum of the first *n* integers as a function of *n*.

b. Solve simple polynomial equations and use technology to approximate solutions for more complex polynomial equations.

D5 Perform operations on polynomial expressions.

a. Add, subtract, multiply, and factor polynomials.

b. Divide one polynomial by a lower-degree polynomial.

c. Know and use the binomial expansion theorem.

d. Relate the expansion of $(a + b)^n$ to the possible outcomes of a binomial experiment and the n^{th} row of Pascal's triangle.

E. Radical and Rational Functions and Equations

Radical equations and functions are a natural outgrowth of power functions. Simple rational equations and functions build on the reciprocal functions of the form $f(x) = \frac{k}{x}$ studied earlier and are directly related to linear and quadratic polynomials and their reciprocals. Algebraic facility with radical and rational forms should be seen as extending earlier work with numeric fractions in upper elementary and middle school.

Successful students will:

E1 Use factoring to reduce rational expressions that consist of the quotient of two simple polynomials.

E2 Perform operations on simple rational expressions.

Simple rational expressions are those whose denominators are linear or quadratic polynomial expressions.

a. Add subtract, multiply, and divide rational expressions having monomial or binomial denominators.

b. Rewrite complex fractions composed of simple rational expressions as a simple fraction in lowest terms.

Example: $\dfrac{(a+b)}{\left(\frac{1}{a}+\frac{1}{b}\right)} = \dfrac{(a+b)}{\frac{(b+a)}{ab}} = (a+b)\bullet\dfrac{ab}{(b+a)} = ab$.

E3 Solve simple rational and radical equations in one variable.

a. Use algebraic, numerical, graphical, and/or technological means to solve rational equations.

b. Use algebraic, numerical, graphical, and/or technological means to solve equations involving a radical.

c. Know which operations on an equation produce an equation with the same solutions and which may produce an equation with fewer or more solutions (lost or extraneous roots) and adjust solution methods accordingly.

E4 Graph simple rational and radical functions in two variables.

a. Graph simple rational functions in two variables; identify the domain, range, intercepts, zeros, and asymptotes of the graph.

b. Graph simple radical functions in two variables; identify the domain, range, intercepts, and zeros of the graph.

c. Relate the algebraic properties of a rational or radical function to the geometric properties of its graph.

Examples: The graph of $y = \frac{x-2}{x^2-1}$ has vertical asymptotes at $x = 1$ and $x = -1$, while the graph of $y = \frac{x-2}{x^2-4}$ has a vertical asymptote at $x = -2$ but a hole at (2, ¼); the graph of $y = \sqrt{x+5} - 2$ is the same as the graph of $y = \sqrt{x}$ translated five units to the left and 2 units down.

F. Function Prototypes and Transformations

Having now added power, polynomial, rational, and radical functions to the classes, or families, of functions studied earlier—linear, direct proportional, reciprocal, exponential, and quadratic—students should begin to understand how functions behave as a class of mathematical objects. They should be able to identify or create prototypes for most function families—determining a prototype for polynomials presents difficulties—and should be able to identify or describe the effect of certain transformations on each function, seeing that a specific type of transformation affects each class of functions in a similar way.

Successful students will:

F1 Analyze exponential functions and relate key characteristics in their algebraic and graphical representations.

a. Describe key characteristics of the graphs of exponential functions and relate these to the coefficients in the general form $f(x) = ab^x + c$ for $b > 0$, $b \neq 1$.

Examples: Know that, if $b > 1$, exponential functions are increasing and that they approach a lower limit if $a > 0$ and an upper limit if $a < 0$ as x decreases; know that, if $0 < b < 1$, exponential functions are decreasing and that they approach a lower limit if $a > 0$ and an upper limit if $a < 0$ as x increases.

F2 Distinguish between the graphs of simple exponential and power functions by their key characteristics.

Be aware that it can be very difficult to distinguish graphs of these various types of functions over small regions or particular subsets of their domains. Sometimes the context of an underlying situation can suggest a likely type of function model.

a. Decide whether a given exponential or power function is suggested by the graph, table of values, or underlying context of a problem.

b. Distinguish between the graphs of exponential growth functions and those representing exponential decay.

c. Distinguish among the graphs of power functions having positive integral exponents, negative integral exponents, and exponents that are positive unit fractions ($f(x) = x^{1/n}$, $n \geq 0$).

d. Identify and explain the symmetry of an even or odd power function.

e. Where possible, determine the domain, range, intercepts, asymptotes, and end behavior of exponential and power functions.

Range is not always possible to determine with precision. End behavior refers to the trend of the graph as x approaches $\pm\infty$.

F3 Distinguish among linear, exponential, power, polynomial, and rational expressions, equations, and functions by their symbolic form.

a. Use the position of the variable in an expression to determine the classification of the expression.

Examples: $f(x) = 3^x$ is an exponential function because the variable is in the exponent, while $f(x) = x^3$ has the variable in the position of a base and is a power function; $f(x) = x^3 - 5$ is a polynomial function but not a power function because of the added constant.

b. Identify or determine a prototypical representation for each family of functions.

Examples: $f(x) = x^2$ is a prototype for quadratic functions; $g(x) = \frac{1}{x}$ is a prototypical reciprocal function.

F4 Explain, illustrate, and identify the effect of simple coordinate transformations on the graphs of power, polynomial and exponential functions; compare these to transformations occasioned by changes in parameters in linear, direct proportional and reciprocal functions.

a. Interpret the graph of $y = f(x - a)$ as the graph of $y = f(x)$ shifted $|a|$ units to the right ($a > 0$) or the left ($a < 0$).

b. Interpret the graph of $y = f(x) + a$ as the graph of $y = f(x)$ shifted $|a|$ units up ($a > 0$) or down ($a < 0$).

c. Interpret the graph of $y = f(ax)$ as the graph of $y = f(x)$ expanded horizontally by a factor of $\frac{1}{|a|}$ if $0 < |a| < 1$ or compressed horizontally by a factor of $|a|$ if $|a| > 1$ and reflected over the y-axis if $a < 0$.

d. Interpret the graph of $y = af(x)$ as the graph of $y = f(x)$ compressed vertically by a factor of $\frac{1}{|a|}$ if $0 < |a| < 1$ or expanded vertically by a factor of $|a|$ if $|a| > 1$ and reflected over the x-axis if $a < 0$.

G. Functions Operations and Logarithms

[OPTIONAL ENRICHMENT UNIT]

If time permits, the basic families of functions can be extended through a unit that introduces logarithmic functions. The unit begins by formalizing operations on functions including function composition. While addition, subtraction, multiplication, and division of functions follow directly from earlier work, the new operation, function composition, permits the definition of a new class of functions. Logarithms are defined as the inverses of exponential functions. This offers an opportunity to revisit and reinforce the understanding of exponential functions originally introduced in Algebra I. The introduction of logarithms permits new solution strategies for exponential equations, encouraging expanded applications including those involving continuous growth or decay.

Successful students will:

G1 Perform operations on simple functions; identify any necessary restrictions on the domain.

a. Determine the sum, difference, product, quotient, and composition of simple functions.

b Analyze the transformations of a function from its graph, formula, or verbal description.

Example: The graph of $f(x) = -3x^2 + 4$ is a vertical dilation by a factor of 3 of the prototype $f(x) = x^2$ followed by a reflection over the x-axis and a translation 4 units up. The resulting vertex of the parabola (0, 4) reflects these transformations and is evident when $f(x) = -3x^2 + 4$ is compared to the vertex form of a parabola $f(x) = a(x - h)^2 + k$.

G2 Analyze characteristics of inverse functions.

a. Identify and explain the relationships among the identity function, composition of functions, and the inverse of a function, along with implications for the domain.

b. Identify the conditions under which the inverse of a function is a function.

c. Determine whether two given functions are inverses of each other.

d. Explain why the graph of a function and its inverse are reflections of one another over the line $y = x$.

G3 Determine the inverse of linear and simple non-linear functions, including any necessary restrictions on the domain.

a. Determine the inverse of a simple polynomial or simple rational function.

b. Identify a logarithmic function as the inverse of an exponential function.

If $x^y = z$, $x > 0$, $x \neq 1$, and $z > 0$, then y is the logarithm to the base x of z. The logarithm $y = \log_x z$ is one of three equivalent forms of expressing the relation $x^y = z$ (the other being $x = \sqrt[y]{z}$).

c. Determine the inverse of a given exponential or logarithmic function.

Example: If $5^a = b$, then $\log_5(b) = a$.

G4 Apply properties of logarithms to solve equations and problems and to prove theorems.

a. Know and use the definition of logarithm of a number and its relation to exponents.

Examples: $\log_2 32 = \log_2 2^5 = 5$; if $x = \log_{10} 3$, then $10^x = 3$ and vice versa.

b. Prove basic properties of logarithms using properties of exponents (or the inverse exponential function).

c. Use properties of logarithms to manipulate logarithmic expressions in order to extract information.

d. Use logarithms to express and solve problems.

Example: Explain why the number of digits in the binary representation of a decimal number N is approximately the logarithm to base 2 of N.

e. Solve logarithmic equations; use logarithms to solve exponential equations.

Examples: $log(x - 3) + log(x - 1) = 0.1$; $5^x = 8$.

H. Statistical Studies and Models

Now that students have deepened their experience with many types of functions and with the effect of transformations on them, they would benefit from engaging in a project that requires collecting and analyzing data. This unit provides students with the opportunities they will need to understand the differences among the major types of statistical studies. For the purposes of applying what they have learned about functions, a project that generates bivariate data would be effective. Included here is an optional enrichment unit that provides students with the opportunity to make meaningful connections between functions, modeling, and data analysis.

Successful students will:

H1 Describe the nature and purpose of sample surveys, experiments, and observational studies, relating each to the types of research questions they are best suited to address.

a. Identify specific research questions that can be addressed by different techniques for collecting data.

b. Critique various methods of data collection used in analyzing real-world problems, such as a clinical trial in medicine, an opinion poll or a report on the effect of smoking on health.

c. Explain why observational studies generally do not lead to good estimates of population characteristics or cause-and-effect conclusions regarding treatments.

H2 Plan and conduct sample surveys, observational studies, and experiments.

a. Recognize and explain the rationale for using randomness in research designs; distinguish between random sampling from a population in sample surveys and random assignment of treatments to experimental units in an experiment.

Random sampling is how items are selected from a population so that the sample data can be used to estimate characteristics of the population; random assignment is how treatments are assigned to experimental units so that comparisons among the treatment groups can allow cause-and-effect conclusions to be made.

b Use simulations to analyze and interpret key concepts of statistical inference.

Key concepts of statistical inference include margin of error and how it relates to the design of a study and to sample size; confidence interval and how it relates to the margin of error; and p-value and how it relates to the interpretation of results from a randomized experiment.

H3 Identify an appropriate family of functions as a model for real data.

[OPTIONAL ENRICHMENT UNIT]

a. Analyze and compare key characteristics of different families of functions; identify prototypical functions as potential models for given data.

b. Apply transformations of data for the purpose of "linearizing" a scatter plot that exhibits curvature.

Examples: Apply squaring, square root, and reciprocal and logarithmic functions to input data, output data, or both; evaluate which transformation produces the strongest linear trend.

c. Use and interpret a residual plot of the relationship among the standard deviation, correlation coefficients and slope of the line to evaluate the goodness of fit of a regression line to transformed data.

d. Estimate the rate of exponential growth or decay by fitting a regression model to appropriate data transformed by logarithms.

e. Estimate the exponent in a power model by fitting a regression model to appropriate data transformed by logarithms.

f. Analyze how linear transformations of data affect measures of center and spread, the slope of a regression line and the correlation coefficient.